Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

ELSEVIER

Journal of Computational and Applied Mathematics 162 (2004) 483–496

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Cubature formulae for spheres, simplices and balls[☆]

Guergana Petrova

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

Received 13 February 2003; received in revised form 8 August 2003

Abstract

We obtain in explicit form the unique Gaussian cubature for balls (spheres) in $\mathbb{R}^n$ based on integrals over balls (spheres), centered at the origin, that integrates exactly all m-harmonic functions. In particular, this formula is exact for all polynomials in n variables of degree $2m-1$. A Gaussian cubature for simplices is also constructed. Upper bounds for the errors for certain smoothness classes are derived.

MSC: 65D32; 65D30; 41A55

Keywords: Gaussian cubature formulae; Polyharmonic functions; Polyharmonic degree of precision

1. Introduction

The classical approach for recovery of a function (or functionals of it) is based on the knowledge of the function values at a set of points. However, as mentioned in [5] in the one-dimensional case, sometimes it is convenient to interpret the function values $\{f(t_j)\}$ as a mean value of f over intervals containing the nodes $\{t_j\}$, because in practice the data $\{f(t_j)\}$ is usually available up to a certain accuracy and even the nodes $\{t_j\}$ may be given approximately (similar interpretation can be given in higher dimensions). Moreover, in some practical problems, different functionals, not necessarily point evaluations, are the only data available and in such cases generalizations of the existing theory and algorithms are required.

Recently, some progress in this direction has been made in the theory of cubature formulae. The extensions involve explicit construction of multivariate cubature, based on various type of information. These problems are extremely difficult. Even though there is a vast literature on cubatures/quadratures

☆ This research was supported in part by the National Science Foundation Grant DMS 0296020.
E-mail address: gpetrova@math.tamu.edu (G. Petrova).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.08.036

(see, for example, [7,16,17] and the references therein) that use as recovery information point values, there are only a few known Gaussian formulae in explicit form that are based on different type of data and integrate exactly all polynomials in n variables of degree as high as possible. Here, we list some of the recent results. Cubature formulae of the form

$$\int_B u(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx \sum_{j=0}^{m-1} C_j \int_{S(r_j)} u(\xi)\,\mathrm{d}\sigma(\xi), \tag{1.1}$$

where B is the unit ball and $S(r_j)$ are the spheres with radius r_j, centered at the origin, have been considered and completely described in [4]. It has been proved [14] that this is the only cubature that integrates exactly all polyharmonic functions of degree $2m$ and that there is no such formula, that is exact for every $(2m+1)$-harmonic function. This cubature is called Gaussian and its nodes and weights are derived in [4] in explicit form. Gaussian quadrature for the unit ball in $\mathbb{R}^n$, based on the Radon projections of the function, was obtained and completely characterized in [6]. In [5], the uniqueness of the Gaussian interval quadrature formula, based on integrals over nonoverlapping subintervals with equal lengths was shown. Such formulae have been computed for particular choices of the weight and the interval.

In the last few years, an approach based on the theory of polyharmonic functions was used to derive cubatures, exact for multivariate algebraic polynomials (see [2–4,8,13]). The technique utilizes the fact that every polynomial in n variables of degree $2m-1$ is a polyharmonic function of order m, and therefore any approximation rule that applies to polyharmonic functions would apply to the corresponding set of algebraic polynomials.

Here, we use this approach to investigate cubature with weight μ of the form

$$\int_{\mathscr{D}(r)} \mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx \sum_{j=0}^{m-1} C_j(r) \int_{\mathscr{D}(r_j)} \mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{1.2}$$

where $r, r_j,\ j=0,\dots,m-1$, is a given set of distinct radii and $\{\int_{\mathscr{D}(r_j)} \mu u\}_{j=0}^{m-1}$ are given data. The domain of integration $\mathscr{D}(r)$ is either a sphere or a ball in $\mathbb{R}^n$, $n \geqslant 1$, centered at the origin with radius r, or a simplex. For $n>1$, we construct explicitly the unique cubature of this type, that integrates exactly all m-harmonic functions (and therefore all algebraic polynomials in n variables of degree $2m-1$). We show that there are no cubatures (1.2) with precision higher than m. We also derive the unique Gaussian formula for the simplex that is exact for all elements in the space $\pi_{m-1}(\mathbb{R}^n)$, consisting of all algebraic polynomials in n variables of degree $m-1$.

For $n=1$, formula (1.2) has the form

$$\int_{-r}^{r} v(t)f(t)\,\mathrm{d}t \approx \sum_{j=0}^{m-1} D_j(r) \int_{-r_j}^{r_j} v(t)f(t)\,\mathrm{d}t. \tag{1.3}$$

It is shown (see Section 4.1) that there is a unique quadrature of this type that is exact for all polynomials of degree $2m-1$, and that this is the highest possible precision. The weights $\{D_j(r)\}$ are calculated for particular weight functions v. Upper bounds for the errors of (1.2) for certain smoothness classes are also obtained.

We would like to mention that, aside of the theory of cubatures, formulae (1.2) can be viewed as extensions of the Pizzetti formula for polyharmonic functions (see [15,11,2]),

$$\int_{B(r)} u(\mathbf{x})\,\mathrm{d}\mathbf{x} = \pi^{n/2} r^n \sum_{k=0}^{m-1} \frac{r^{2k}}{2^{2k}\Gamma(n/2+k+1)} \frac{\Delta^k u(0)}{k!},$$

and its analogue on the sphere [11].

2. Preliminaries and notation

Let $\mathscr{D}$ be a simply connected domain in $\mathbb{R}^n, B(r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = (\sum_{i=1}^n x_i^2)^{1/2} < r\}$ be the ball with radius r, centered at the origin, and $S(r) := \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = r\}$ be its boundary (we will omit writing r in the case $r = 1$). A function u, defined on $\mathscr{D} \subset \mathbb{R}^n$, is called a polyharmonic function of order m (or m-harmonic function) (see [1,11]) if $u \in C^{2m-1}(\bar{\mathscr{D}}) \cap C^{2m}(\mathscr{D})$ ($\bar{\mathscr{D}}$ denotes the closure of $\mathscr{D}$) and it satisfies the equation

$$\Delta^m u(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathscr{D}, \quad \text{where } \Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \ \Delta^m := \Delta \Delta^{m-1}. \tag{2.1}$$

In particular, when $m = 1$ ($m = 2$), u is called harmonic (biharmonic). The set of all m-harmonic functions on $\mathscr{D}$ is denoted by $H^m(\mathscr{D})$. Different representations of m-harmonic functions are available. Here, we shall use the lemma (see [4, Lemma 2]):

Lemma 2.1. *Let $\phi_0, \dots, \phi_{m-1}$, be a basis in the space of univariate algebraic polynomials of degree $m-1$. For each $u \in H^m(B(r))$ there exist unique functions $b_0, \dots, b_{m-1}$, each harmonic in $B(r)$, such that*

$$u(\mathbf{x}) = \sum_{j=0}^{m-1} \phi_j(|\mathbf{x}|^2) b_j(\mathbf{x}), \quad \mathbf{x} \in B(r). \tag{2.2}$$

The particular choice of $\phi_j(t) = t^j$ recovers the Almansi's expansion (see [1, Proposition 1.3, p. 4]). It is clear that

$$\pi_{2m-1}(\mathbb{R}^n) \subset H^m(B(r)). \tag{2.3}$$

We shall also employ the fact that for every harmonic function b

$$\int_{S(r)} b(\xi)\,\mathrm{d}\sigma(\xi) = \gamma_n r^{n-1} b(0), \quad \gamma_n = \frac{n\pi^{n/2}}{\Gamma(n/2+1)}, \tag{2.4}$$

where $\mathrm{d}\sigma$ is the $(n-1)$-dimensional surface measure on $S(r)$, and Γ is the Gamma function.

We are interested in cubature (1.2) that are exact for all polyharmonic functions on $\mathscr{D}(r)$ ($\mathscr{D}(r)$ being $B(r)$ or $S(r)$) of order as high as possible. If p is the largest natural number for which (1.2) is exact for all $u \in H^p(\mathscr{D}(r))$, we call p a polyharmonic degree of precision (PDP) of (1.2). Recall, that in the classical case, we say that (1.3) has algebraic degree of precision (ADP) p, if it integrates exactly all elements in $\pi_p(\mathbb{R})$, and there is a polynomial in $\pi_{p+1}(\mathbb{R})$ for which the quadrature is not exact. Formulae for numerical integration with the best possible PDP(ADP) are called Gaussian cubatures (quadratures). The error of (1.2) for a particular function f is denoted by

$$\begin{aligned}\mathscr{E}_m(f) &= \mathscr{E}_m(\mathscr{D}(r),\mathscr{D}(r_0),\ldots,\mathscr{D}(r_{m-1}),f)\\ &:= \left|\int_{\mathscr{D}(r)}\mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x} - \sum_{j=0}^{m-1}C_j(r)\int_{\mathscr{D}(r_j)}\mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x}\right|.\end{aligned}$$

Then, the error for a function class $\mathscr{H}$ is

$$\mathscr{E}_m(\mathscr{H}) = \mathscr{E}_m(\mathscr{D}(r),\mathscr{D}(r_0),\ldots,\mathscr{D}(r_{m-1}),\mathscr{H}) := \sup_{f\in\mathscr{H}}\mathscr{E}_m(\mathscr{D}(r),\mathscr{D}(r_0),\ldots,\mathscr{D}(r_{m-1}),f). \tag{2.5}$$

We derive an upper bound for (2.5) in the case of $\mathscr{D}=B$ and smoothness class $\mathscr{W}_\alpha$. We use the theorem (see [10, Theorem 1])

$$E_m(f,\bar{B})_\infty \leqslant Cm^{-\alpha} \Leftrightarrow f\in\mathscr{W}_\alpha. \tag{2.6}$$

Here

$$E_m(f,\overline{\mathscr{D}(R)})_\infty := \inf_{P\in\pi_m(\mathbb{R}^n)}\|f-P\|_{L_\infty(\overline{\mathscr{D}(R)})} \tag{2.7}$$

is the error of best polynomial approximation of f in $L_\infty(\overline{D(R)})$ and $\mathscr{W}_\alpha, 0<\alpha<1$, is the smoothness class

$$\mathscr{W}_\alpha := \left\{ f : |f(\mathbf{x})-f(\mathbf{y})| \leqslant C\left(|\mathbf{x}-\mathbf{y}| + \frac{||\mathbf{x}|-|\mathbf{y}||}{\sqrt{1-|\mathbf{x}|^2}+\sqrt{1-|\mathbf{y}|^2}}\right)^\alpha, \quad \mathbf{x},\mathbf{y}\in\bar{B}\right\}. \tag{2.8}$$

In our error estimates, we employ also the Lebesgue function Λ_{m-1} of order $m-1$ for the nodes $\mathscr{T} := (t_0,\ldots,t_{m-1})$, which is

$$\Lambda_{m-1}(t,\mathscr{T}) := \sum_{j=0}^{m-1}|\ell_j(t,\mathscr{T})|, \tag{2.9}$$

with $\ell_j(t,\mathscr{T}),\ j=0,\ldots,m-1$, being the basic Lagrange polynomials for the nodes $\mathscr{T}$.

3. Cubature formulae for the sphere in $\mathbb{R}^n$

In this section, we investigate cubature of the form

$$\int_{S(r)}u(\xi)\,\mathrm{d}\sigma(\xi) \approx \sum_{j=0}^{m-1}A_j(r)\int_{S(r_j)}u(\xi)\,\mathrm{d}\sigma(\xi), \quad r\neq r_j, \tag{3.1}$$

for any distinct radii $\{r, r_j\}_{j=0}^{m-1}$. Let $R := \max_{j=0,\dots,m-1}\{r, r_j\}$. Formula (3.1) is not exact for the polynomial

$$L(\mathbf{x}) := \prod_{j=0}^{m-1}(|\mathbf{x}|^2 - r_j^2) \in \pi_{2m}(\mathbb{R}^n) \subset H^{m+1}(B(R)),$$

and hence the PDP(3.1) $\leqslant m$.

We construct the unique formula of type (3.1), that is exact for all $u \in H^m(B(R))$. In particular, this cubature will have ADP $= 2m-1$ (PDP $= m$), and can be viewed as a multidimensional analogue of the Gaussian quadrature in the one-dimensional case. More precisely, the following theorem is true:

Theorem 3.1. *For any set of distinct radii* $\{r, r_j\}_{j=0}^{m-1}$ *and information* $\{\int_{S(r_j)} u(\xi)\,\mathrm{d}\sigma(\xi)\}_{j=0}^{m-1}$, *there is a unique cubature formula*

$$\int_{S(r)} u(\xi)\,\mathrm{d}\sigma(\xi) \approx \sum_{j=0}^{m-1} A_j(r) \int_{S(r_j)} u(\xi)\,\mathrm{d}\sigma(\xi), \tag{3.2}$$

exact for all $u \in H^m(B(R))$, *where* $R := \max_{j=0,\dots,m-1}\{r, r_j\}$. *Its weights* $A_j(r)$ *are*

$$A_j(r) = \frac{r^{n-1}}{r_j^{n-1}}\,\ell_j(r^2), \tag{3.3}$$

with ℓ_j, $j = 0, \dots, m-1$, *being the basic Lagrange polynomials for the nodes* $r_0^2, \dots, r_{m-1}^2$.

Proof. Lemma 2.1 (with $\phi_j = \ell_j$), gives

$$u(\mathbf{x}) = \sum_{j=0}^{m-1} \ell_j(|\mathbf{x}|^2) b_j(\mathbf{x}), \quad \text{where } \ell_j(r_k^2) = \delta_{jk}. \tag{3.4}$$

Integration of (3.4) over $S(r_k)$ and application of (2.4) results in

$$\int_{S(r_k)} u(\xi)\,\mathrm{d}\sigma(\xi) = \sum_{j=0}^{m-1} \ell_j(r_k^2) \int_{S(r_k)} b_j(\xi)\,\mathrm{d}\sigma(\xi) = \gamma_n r_k^{n-1} b_k(0),$$

and therefore

$$b_k(0) = \frac{1}{\gamma_n r_k^{n-1}} \int_{S(r_k)} u(\xi)\,\mathrm{d}\sigma(\xi). \tag{3.5}$$

Further, we integrate (3.4) over the sphere $S(r)$ and get

$$\int_{S(r)} u(\xi)\,\mathrm{d}\sigma(\xi) = \sum_{j=0}^{m-1} \ell_j(r^2) \int_{S(r)} b_j(\xi)\,\mathrm{d}\sigma(\xi),$$

which together with (2.4) and (3.5) leads to

$$\int_{S(r)} u(\xi)\,\mathrm{d}\sigma(\xi) = \sum_{j=0}^{m-1} \ell_j(r^2)\gamma_n r^{n-1} b_j(0) = \sum_{j=0}^{m-1} \ell_j(r^2)\frac{r^{n-1}}{r_j^{n-1}} \int_{S(r_j)} u(\xi)\,\mathrm{d}\sigma(\xi)$$

$$= \sum_{j=0}^{m-1} A_j(r) \int_{S(r_j)} u(\xi)\,\mathrm{d}\sigma(\xi)$$

with weights $A_j(r)$ given by (3.3). The existence is proved.

Let (3.1) be exact for all elements in $H^m(B(R))$. It is exact, in particular, for $\ell_k(|\mathbf{x}|^2) \in \pi_{2m-1}(\mathbb{R}^n)$, $k = 0, \ldots, m-1$. We apply (3.1) to $\ell_k(|\mathbf{x}|^2)$ and since

$$\int_{S(r_j)} \ell_k(|\mathbf{x}|^2)\,\mathrm{d}\sigma(\xi) = \delta_{kj}\gamma_n r_j^{n-1}, \qquad \int_{S(r)} \ell_k(|\mathbf{x}|^2)\,\mathrm{d}\sigma(\xi) = \gamma_n r^{n-1}\ell_k(r^2),$$

we arrive at $A_j(r) = (r^{n-1}/r_j^{n-1})\ell_j(r^2)$, $j = 0, \ldots, m-1$. The proof is completed. □

Remark 3.2. An approach similar to [4], can be applied to construct cubature formulae where the information available are the integrals of u and its consecutive normal derivatives $\frac{\partial^k}{\partial v^k} u$ of any given multiplicities $v_0, \ldots, v_{m-1}$, namely

$$\int_{S(r)} u(\xi)\,\mathrm{d}\sigma(\xi) \approx \sum_{j=0}^{m-1} \sum_{k=0}^{v_j-1} C_{jk}(r) \int_{S(r_j)} \frac{\partial^k}{\partial v^k} u(\xi)\,\mathrm{d}\sigma(\xi).$$

4. Cubature formulae for the ball in $\mathbb{R}^n$

In this section, we consider the cubature

$$\int_{B(r)} \mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x} \approx \sum_{j=0}^{m-1} C_j(r) \int_{B(r_j)} \mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{4.1}$$

with strictly positive weight μ, $\mu(t) > 0$, $t \neq 0$. There is a one-to-one correspondence between cubature (4.1) and the univariate interval quadrature (1.3). Properties of (1.3) are inherited in (4.1). Therefore, investigation of (4.1) requires a complete understanding of interval quadrature. We study the latter formulae in the next section.

4.1. Interval cubature formulae

Here, we investigate the interval quadratures (1.3) with strictly positive even weight v, $v(t) > 0$, $t \neq 0$. In what follows, we use the next lemma, which we prove for completeness:

Lemma 4.1. *For every $0 < r_0 < \cdots < r_{m-1}$ and strictly positive even function v, the determinant* $\det\{\int_{-r_j}^{r_j} v(t)t^{2k}\,\mathrm{d}t\} \neq 0$.

Proof. Assume the opposite. Then, there is a nontrivial polynomial $Q \in \pi_{m-1}(\mathbb{R})$, for which

$$\int_0^{r_j} v(t)Q(t^2)\,\mathrm{d}t = 0, \quad j = 0, \ldots, m-1,$$

and therefore there are ξ_j's, $j = 0, \ldots, m-1$, $\xi_j \in (r_{j-1}, r_j)$, such that

$$Q(\xi_j^2)\int_{r_{j-1}}^{r_j} v(t)\,\mathrm{d}t = \int_{r_{j-1}}^{r_j} v(t)Q(t^2)\,\mathrm{d}t = 0, \quad r_{-1} := 0, \quad j = 0, \ldots, m-1,$$

with $v(t) > 0$, $t \neq 0$. Hence, the nontrivial polynomial $Q \in \pi_{m-1}(\mathbb{R})$ has at least m zeroes in $(0, \infty)$ (at least one zero in each of (r_{j-1}^2, r_j^2), $j = 0, \ldots, m-1$)—a contradiction. □

Further, we show that for every set of nodes r_j, $j = 0, \ldots, m-1$, there is only one Gaussian quadrature of type (1.3). Its $\mathrm{ADP} = 2m-1$ and its weights can be computed explicitly. We give the formula for the weights in the case $v(t) = |t|^s$. More precisely, the following theorem is true:

Theorem 4.2. *For every set of distinct radii $\{r, r_j\}_{j=0}^{m-1}$ and every strictly positive even weight v there is a unique quadrature of type (1.3) with $\mathrm{ADP} = 2m-1$. Its weights are*

$$D_j(r) = 2\int_0^r v(t)\bar{\ell}_j(t^2)\,\mathrm{d}t, \tag{4.2}$$

where $\bar{\ell}_j \in \pi_{m-1}(\mathbb{R})$ are uniquely determined by the interpolation conditions

$$2\int_0^{r_j} v(t)\bar{\ell}_k(t^2)\,\mathrm{d}t = \delta_{jk}, \quad j,k = 0, \ldots, m-1.$$

In the special case of $v(t) = |t|^s$, s—constant,

$$D_j(r) = \frac{r^{s+1}}{r_j^{s+1}}\,\ell_j(r^2), \quad j = 0, \ldots, m-1,$$

with $\{\ell_j\}$ being the basic Lagrange polynomials for the nodes $r_0^2, \ldots, r_{m-1}^2$. There is no quadrature of type (1.3) with $\mathrm{ADP} \geqslant 2m$.

Proof. We consider quadrature (1.3) with fixed nodes $\{r_j\}$. We will construct a polynomial $Q_0 \in \pi_{2m}(\mathbb{R})$ for which (1.3) is not exact, and therefore $\mathrm{ADP}(1.3) \leqslant 2m-1$.

First, we show that the interpolation problem (see [12])

$$\int_{r_{k-1}}^{r_k} v(t)Q(t)\,\mathrm{d}t = 0, \quad k = 0, \ldots, m-1, \tag{4.3}$$

in the set of all even algebraic polynomials of degree $2m$ has a nontrivial solution Q_0. Indeed, there is a nontrivial vector $(c_0, c_1, \ldots, c_m) \in \mathbb{R}^{m+1}$ that is perpendicular to the space $span\{\Lambda_0, \ldots, \Lambda_{m-1}\}$, where

$$\Lambda_k := \left(\int_{r_{k-1}}^{r_k} v(t)\,\mathrm{d}t, \int_{r_{k-1}}^{r_k} v(t)t^2\,\mathrm{d}t, \ldots, \int_{r_{k-1}}^{r_k} v(t)t^{2(m-1)}\,\mathrm{d}t, \int_{r_{k-1}}^{r_k} v(t)t^{2m}\,\mathrm{d}t \right) \in \mathbb{R}^{m+1},$$

that is

$$\sum_{j=0}^{m} c_j \int_{r_{k-1}}^{r_k} v(t)t^{2j}\,\mathrm{d}t = 0, \quad k = 0, \ldots, m-1. \tag{4.4}$$

Relation (4.4) can be written as

$$\int_{r_{k-1}}^{r_k} v(t)Q_0(t)\,\mathrm{d}t = 0, \quad k = 0, \ldots, m-1,$$

where Q_0 is the polynomial $Q_0(t) := \sum_{j=0}^{m} c_j t^{2j}$, and thus is the nontrivial solution of problem (4.3). Now, we show that (1.3) is not exact for Q_0. Assume the opposite. Since vQ_0 is even function, we have

$$\int_{-r_k}^{r_k} v(t)Q_0(t)\,\mathrm{d}t = 2\int_{0}^{r_k} v(t)Q_0(t)\,\mathrm{d}t$$
$$= 2\sum_{j=0}^{k} \int_{r_{j-1}}^{r_j} v(t)Q_0(t)\,\mathrm{d}t = 0, \quad k = 0, \ldots, m-1, \quad r_{-1} := 0. \tag{4.5}$$

Relation (4.5) and quadrature (1.3), applied to Q_0, result in

$$\int_{-r}^{r} v(t)Q_0(t)\,\mathrm{d}t = 0.$$

The above equation and (4.5) give (we assume $r_0 < \cdots < r_{m-1}$)

$$\int_{r_{l-1}}^{r} v(t)Q_0(t)\,\mathrm{d}t = 0 = \int_{r}^{r_l} v(t)Q_0(t)\,\mathrm{d}t, \quad \text{if } r_{l-1} < r < r_l,$$

for some $l \in \{0, \ldots, m-1\}$, or

$$\int_{r_{m-1}}^{r} v(t)Q_0(t)\,\mathrm{d}t = 0, \quad \text{if } r_{m-1} < r.$$

As in the proof of Lemma 4.1, this implies that Q_0 has at least $m+1$ zeroes on $(0, \infty)$ (at least one zero in each of the intervals $(0, r_0), (r_0, r_1), \ldots, (r_{l-1}, r), (r, r_l) \ldots, (r_{m-2}, r_{m-1})$, if $r_{l-1} < r < r_l$, or at least one zero in each of $(0, r_0), (r_0, r_1), \ldots, (r_{m-2}, r_{m-1}), (r_{m-1}, r)$, if $r_{m-1} < r$). Since Q_0 is even, it then has at least $2m+2$ zeros. This contradicts the fact that Q_0 is a nontrivial polynomial of degree $2m$.

Next, we construct the quadrature of type (1.3) with ADP $=2m-1$. It follows from Lemma 4.1 that the interpolation problem of finding $p\in\pi_{m-1}(\mathbb{R})$, such that

$$2\int_0^{r_k} v(t)p(t^2)\,\mathrm{d}t=\gamma_k,\quad k=0,\dots,m-1, \tag{4.6}$$

has a unique solution for any choice of data $\{\gamma_k\}$. The solution is

$$p(t^2)=\sum_{j=0}^{m-1}2\bar{\ell}_j(t^2)\int_0^{r_j} v(t)p(t^2)\,\mathrm{d}t=\sum_{j=0}^{m-1}\gamma_j\bar{\ell}_j(t^2), \tag{4.7}$$

where $\bar{\ell}_j$ are the polynomials satisfying (4.6) with $\gamma_k=\delta_{jk}$. We integrate (4.7) and get

$$\begin{aligned}\int_{-r}^{r} v(t)p(t^2)\,\mathrm{d}t&=\sum_{j=0}^{m-1}\left[\int_{-r}^{r} v(t)\bar{\ell}_j(t^2)\,\mathrm{d}t\right]\left[2\int_0^{r_j} v(t)p(t^2)\,\mathrm{d}t\right]\\&=\sum_{j=0}^{m-1}D_j(r)\int_{-r_j}^{r_j} v(t)p(t^2)\,\mathrm{d}t,\quad\text{where } D_j(r)=2\int_0^{r} v(t)\bar{\ell}_j(t^2).\end{aligned} \tag{4.8}$$

Since $\int_{-a}^{a} v(t)f(t)\,\mathrm{d}t=0$ for every odd f and every constant a, formula (4.8) is exact not only for $p(t^2), p\in\pi_{m-1}(\mathbb{R})$, but also for all polynomials in $\pi_{2m-1}(\mathbb{R})$. Therefore it is Gaussian. The uniqueness follows directly from Lemma 4.1.

According to (4.8), the calculation of $D_j(r)$ requires a knowledge for the corresponding $\bar{\ell}_j$'s. Let $v(t)=|t|^s$, and $\{\ell_j\}\in\pi_{m-1}(\mathbb{R})$ be the basic Lagrange polynomials for the nodes $r_0^2,\dots,r_{m-1}^2$. We define

$$q_j(x):=\frac{1}{2r_j^{s+1}}((s+1)\ell_j(x)+2x\ell_j'(x)),\quad q_j\in\pi_{m-1}(\mathbb{R}).$$

It is clear that for

$$F_j(t):=\frac{t^{s+1}}{r_j^{s+1}}\,\ell_j(t^2),$$

$F_j(0)=0$, $F_j(r_k)=\delta_{jk}$ and $F_j'(t)=2t^s q_j(t^2)$. Then

$$2\int_0^{r_k} t^s q_j(t^2)\,\mathrm{d}t=\int_0^{r_k} F_j'(t)\,\mathrm{d}t=F_k(r_j)-F_k(0)=\delta_{jk},$$

and therefore (because of the uniqueness) $q_j\equiv\bar{\ell}_j$. We calculate the weights $D_j(r)$ as follows:

$$D_j(r)=2\int_0^{r} t^s\bar{\ell}_j(t^2)\,\mathrm{d}t=2\int_0^{r} t^s q_j(t^2)\,\mathrm{d}t=\int_0^{r} F_j'(t)\,\mathrm{d}t=F_j(r)-F_j(0)=\frac{r^{s+1}}{r_j^{s+1}}\,\ell_j(r^2).$$

The proof is completed. □

Theorem 4.3. *Let (1.3) be the Gaussian interval quadrature formula with weight $v(t) = |t|^s$, s—constant. The error of (1.3) for any function f is estimated by*

$$\mathscr{E}_m(f) \leqslant \frac{2R^{s+1}}{s+1} E_{2m-1}(f,[-R,R])_\infty \left(1 + \max_{t\in[0,R^2]} \Lambda_{m-1}(t,\mathscr{T})\right),$$

where $R = \max_{j=0,\dots,m-1}\{r, r_j\}$ and $\mathscr{T} = (r_0^2,\dots,r_{m-1}^2)$.

Proof. Quadrature (1.3) is exact for $\pi_{2m-1}(\mathbb{R})$, and therefore for any function f and any $P\in\pi_{2m-1}(\mathbb{R})$ we have that $\mathscr{E}_m(f) := \mathscr{E}((-r,r),(-r_0,r_0),\dots,(-r_{m-1},r_{m-1}),f)$ is bounded by

$$\begin{aligned}\mathscr{E}_m(f) &\leqslant \int_{-r}^{r} v(t)|f(t)-P(t)|\,\mathrm{d}t + \sum_{j=0}^{m-1}|D_j(r)|\int_{-r_j}^{r_j} v(t)|f(t)-P(t)|\,\mathrm{d}t \\ &\leqslant \frac{2r^{s+1}}{s+1}\|f-P\|_{L_\infty([-R,R])}\left(1+\sum_{j=0}^{m-1}|\ell_j(r^2)|\right) \\ &\leqslant \frac{2R^{s+1}}{s+1}\|f-P\|_{L_\infty([-R,R])}\left(1+\max_{t\in[0,R^2]}\Lambda_{m-1}(t,\mathscr{T})\right).\end{aligned}$$

Here, in the next to the last inequality we have used the explicit form of the coefficients $D_j(r)$ from Theorem 4.2. Using definition (2.7), we complete the proof. □

4.2. Gaussian cubature for the ball

In this section, we explicitly construct the unique Gaussian formula of type (4.1) that integrates exactly all m-harmonic functions and derive an error estimate for the smoothness class $\mathscr{W}_\alpha$ (see (2.8) for the definition of $\mathscr{W}_\alpha$). First, we show in Lemma 4.4 how the PDP of (4.1) can be determined in a quantitative way, and use it to present the one-to-one correspondence between formulae (4.1) and (1.3) (Lemma 4.5). This allows a straightforward application to (4.1) of the theory, developed for interval quadratures (1.3).

Lemma 4.4. *Formula (4.1) integrates exactly all polynomials $|\mathbf{x}|^{2l}, l=0,\dots,p-1$, and is not exact for $|\mathbf{x}|^{2p}$ if and only if* PDP(4.1) $= p$.

Proof. Let (4.1) be exact for $|\mathbf{x}|^{2l}$, $l=0,\dots,p-1$, and does not integrate exactly $|\mathbf{x}|^{2p}$. We apply (4.1) to $|\mathbf{x}|^{2l}, l=0,\dots,p-1$, and from the fact that for any a

$$\int_{B(a)} \mu(|\mathbf{x}|)|\mathbf{x}|^{2l}\,\mathrm{d}\mathbf{x} = \gamma_n\int_0^a \mu(t)t^{2l+n-1}\,\mathrm{d}t,$$

we obtain

$$\int_0^r \mu(t)t^{2l+n-1}\,\mathrm{d}t = \sum_{j=0}^{m-1} C_j(r)\int_0^{r_j}\mu(t)t^{2l+n-1}\,\mathrm{d}t,\quad l=0,\dots,p-1. \tag{4.9}$$

Let u be any polyharmonic function of order p, and therefore has representation (2.2). Integration of (2.2) (with $\Phi(t)=t^j$), and formulae (2.4) and (4.9) give

$$\begin{aligned}\int_{B(r)}\mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \sum_{l=0}^{p-1}\int_0^r \mu(t)\left[t^{2l}\int_{S(t)} b_l(\xi)\,\mathrm{d}\sigma(\xi)\right]\mathrm{d}t = \sum_{l=0}^{p-1}\gamma_n b_l(0)\int_0^r \mu(t)t^{2l+n-1}\,\mathrm{d}t\\
&= \sum_{l=0}^{p-1}\gamma_n b_l(0)\sum_{j=0}^{m-1}C_j(r)\int_0^{r_j}\mu(t)t^{2l+n-1}\,\mathrm{d}t\\
&= \sum_{j=0}^{m-1}C_j(r)\int_0^{r_j}\mu(t)\left[\sum_{l=0}^{p-1}t^{2l}\gamma_n t^{n-1}b_l(0)\right]\mathrm{d}t\\
&= \sum_{j=0}^{m-1}C_j(r)\int_{B(r_j)}\mu(|\mathbf{x}|)u(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad (4.10)\end{aligned}$$

i.e., the cubature integrates exactly all p-harmonic functions. Since (4.1) is not exact for the $(p+1)$-harmonic function $|\mathbf{x}|^{2p}$, we obtain that $\mathrm{PDP}(4.1)=p$.

Next, if $\mathrm{PDP}(4.1)=p$, (2.3) gives that (4.1) is exact for $|\mathbf{x}|^{2l}, l=0,\dots,p-1$. If we assume that the cubature is exact also for $|\mathbf{x}|^{2p}$, then by the above arguments it will integrate all $(p+1)$-harmonic functions, which will be a contradiction. □

Lemma 4.5. $\mathrm{PDP}(4.1)=p$ *if and only if the quadrature* (1.3) *with* $D_j(r)=C_j(r)$ *and*

$$v(t) := \begin{cases}\mu(t)t^{n-1}, & 0<t,\\ \mu(-t)|t|^{n-1}, & t<0,\end{cases} \qquad (4.11)$$

has $\mathrm{ADP}(1.3)=(2p-1)$.

Proof. Let $\mathrm{PDP}(4.1)=p$. Lemma 4.4 gives that (4.1) is exact for $|\mathbf{x}|^{2l}, l=0,\dots,p-1$, and it is not exact for $|\mathbf{x}|^{2p}$. As in Lemma 4.4, we obtain that (4.9) holds, and that

$$\int_0^r \mu(t)t^{n-1}t^{2p}\,\mathrm{d}t \neq \sum_{j=0}^{m-1}C_j(r)\int_0^{r_j}\mu(t)t^{n-1}t^{2p}\,\mathrm{d}t. \qquad (4.12)$$

Integration rules of odd and even functions over symmetric intervals give that (4.9) is equivalent to

$$\int_{-r}^{r} v(t)Q(t)\,\mathrm{d}t = \sum_{j=0}^{m-1}C_j(r)\int_{-r_j}^{r_j} v(t)Q(t)\,\mathrm{d}t, \qquad (4.13)$$

where $Q\in\pi_{2p-1}(\mathbb{R})$, and v is the weight defined in (4.11). Relation (4.12) shows that (4.13) is not exact for $Q(t)=t^{2p}$, and therefore (4.13) is the Gaussian interval quadrature (1.3) with $\mathrm{ADP}=2p-1$.

Now, let (1.3) be the Gaussian interval quadrature with weight v, defined in (4.11). Then (4.9) holds ((1.3) is exact for $t^{2l}, l=0,\dots,p-1$), and as in Lemma 4.4 we get (4.10) for any p-harmonic function, with $C_j(r)$ being $D_j(r)$. As in Lemma 4.4, we show that the obtained cubature is not exact for $|\mathbf{x}|^{2p}$. The proof is completed. □

The next theorem gives the explicit construction of the unique Gaussian cubature of type (4.1).

Theorem 4.6. *Let $r \in (0,1)$, and $\{r_j\}, 0 < r_0 < \cdots < r_{m-1} < 1$, be given radii. Then, there is a unique cubature of type (4.1) that is exact for all $u \in H^m(B)$. The weights $C_j(r)$ are given explicitly by*

$$C_j(r) = 2\int_0^r \mu(t)t^{n-1}\bar{\ell}_j(t^2)\,\mathrm{d}t,$$

where $\bar{\ell}_j \in \pi_{m-1}(\mathbb{R})$ are uniquely determined by the interpolation conditions

$$2\int_0^{r_j} \mu(t)t^{n-1}\bar{\ell}_k(t^2)\,\mathrm{d}t = \delta_{jk}, \quad j,k = 0,\ldots,m-1.$$

There is no formula of this type with PDP $> m$.

For weights $\mu(t) = |t|^s$, s—constant, the coefficients are

$$C_j(r) = \frac{r^{s+n}}{r_j^{s+n}}\,\ell_j(r^2), \quad j = 0,\ldots,m-1,$$

with $\{\ell_j\}$ being the basic Lagrange polynomials for the nodes $r_0^2,\ldots,r_{m-1}^2$. The error of (4.1) for the class $\mathscr{W}_\alpha$ is (see (2.8) for the definition of $\mathscr{W}_\alpha$)

$$\mathscr{E}_m(B, B(r_0),\ldots,B(r_{m-1})) \leqslant \frac{\gamma_n}{s+n}\, m^{-\alpha}\left(1 + \max_{t\in[0,1]} \Lambda_{m-1}(t,\mathscr{T})\right), \tag{4.14}$$

where $\mathscr{T} = (r_0^2,\ldots,r_{m-1}^2)$.

Proof. The theorem follows directly from Lemma 4.5, Theorem 4.2, and relation (2.6). □

Remark 4.7. Theorem 4.6 can be stated for any choice of distinct radii (not necessary all of them being in $(0,1)$). In this case, a proper modification of the class $\mathscr{W}_\alpha$ is needed.

5. Cubature formula for simplices in $\mathbb{R}^n$

In this section, we study formulae of type

$$\int_{\Sigma(r)} \frac{1}{\sqrt{u_1\ldots u_n}}\, f(\mathbf{u})\,\mathrm{d}\mathbf{u} \approx \sum_{j=0}^{m-1} E_j(r)\int_{\Sigma(r_j)} \frac{1}{\sqrt{u_1\ldots u_n}}\, f(\mathbf{u})\,\mathrm{d}\mathbf{u}, \tag{5.1}$$

where $\Sigma(r)$ is the simplex $\Sigma(r) := \{\mathbf{u} \in \mathbb{R}^n : u_1 \geqslant 0,\ldots,u_n \geqslant 0, \sum_{i=0}^n u_i < r^2\}$. We construct the unique formula of this type that integrates exactly all polynomials in $\pi_{m-1}(\mathbb{R}^n)$. We show that there is no such formula with algebraic degree of precision higher than $(m-1)$. The proof is based on the theory we develop in Section 4 and the fact that for any g, and any constant a (see [18])

$$\int_{B(a)} g(x_1^2,\dots,x_n^2)\,\mathrm{d}\mathbf{x} = \int_{\Sigma(a)} \frac{1}{\sqrt{u_1\dots u_n}}\, g(u_1,\dots,u_n)\,\mathrm{d}\mathbf{u}. \tag{5.2}$$

The latter formula establishes the correspondence between cubature over balls and simplices. The following lemma holds:

Lemma 5.1. *Cubature* (4.1) *with* $\mu\equiv 1$ *has* PDP(4.1) $= p$ *if and only if cubature* (5.1) (*with* $E_j(r) = C_j(r)$) *has* ADP(5.1) $= p-1$.

Proof. By Lemma 4.4, PDP(4.1) $= p$ if and only if (4.1) integrates exactly the polynomials $|\mathbf{x}|^{2l}$, $l=0,\dots,p-1$, and is not exact for $|\mathbf{x}|^{2p}$. Therefore, to prove the lemma, it is enough to consider (4.1) only for $|\mathbf{x}|^{2l}$, $l=0,\dots,p$.

Let ADP(5.1) $= p-1$. We apply (5.1) to $(\sum_{i=1}^n u_i)^l \in \pi_{p-1}(\mathbb{R}^n)$, $l=0,\dots,p-1$, and use (5.2) for each of the integrals in the cubature. We derive that

$$\int_{B(r)} |\mathbf{x}|^{2l}\,\mathrm{d}\mathbf{x} = \sum_{j=0}^{m-1} E_j(r)\int_{B(r_j)} |\mathbf{x}|^{2l}\,\mathrm{d}\mathbf{x}, \quad l=0,\dots,p-1. \tag{5.3}$$

If we assume that (5.3) is exact for $|\mathbf{x}|^{2p}$, then again by (5.2), we get that (5.1) is exact for $(u_1+\cdots+u_n)^p$, which contradicts the fact that ADP(5.1) $= p-1$. Hence, by Lemma 4.4, the polyharmonic degree of precision of (4.1) (with $\mu\equiv 1$, and $E_j(r) = C_j(r)$) is p.

Now, let PDP(4.1) $= p$. Then, it will be exact for $S(x_1^2,\dots,x_n^2)$, where S is an arbitrary polynomial in $\pi_{p-1}(\mathbb{R}^n)$, and from (5.2) we get

$$\int_{\Sigma(r)} \frac{1}{\sqrt{u_1\dots u_n}}\, S(\mathbf{u})\,\mathrm{d}\mathbf{u} = \sum_{j=0}^{m-1} C_j(r)\int_{\Sigma(r_j)} \frac{1}{\sqrt{u_1\dots u_n}}\, S(\mathbf{u})\,\mathrm{d}\mathbf{u}, \quad \text{for any } S\in\pi_{p-1}(\mathbb{R}^n). \tag{5.4}$$

Formula (5.4) is not exact for $(u_1+\ldots+u_n)^p$, because otherwise (again using (5.2)) it will follow that (4.1) is exact for $|\mathbf{x}|^{2p}$, which contradicts Lemma 4.4. Therefore the algebraic degree of precision of (5.1) (with $E_j(r) = C_j(r)$) is $p-1$. □

The main result in this section is the following theorem, which is a direct corollary of Lemma 5.1 and Theorem 4.6.

Theorem 5.2. *For every set of distinct radii* $\{r, r_j\}_{j=0}^{m-1}$, *there is a unique cubature*

$$\int_{\Sigma(r)} \frac{1}{\sqrt{u_1\dots u_n}}\, f(\mathbf{u})\,\mathrm{d}\mathbf{u} \approx \sum_{j=0}^{m-1} E_j(r)\int_{\Sigma(r_j)} \frac{1}{\sqrt{u_1\dots u_n}}\, f(\mathbf{u})\,\mathrm{d}\mathbf{u}, \tag{5.5}$$

that is exact for all $f\in\pi_{m-1}(\mathbb{R}^n)$. *The weights* $E_j(r)$ *are given explicitly by*

$$E_j(r) = \frac{r^n}{r_j^n}\,\ell_j(r^2), \quad j=0,\dots,m-1,$$

where ℓ_j *are the basic Lagrange polynomials for the nodes* $r_0^2,\dots,r_{m-1}^2$.

Remark 5.3. As in Theorem 4.6, error estimates for cubature (5.5) and certain function classes can be derived. For this, direct and inverse theorems of polynomial approximation over a convex body need to be used (see, for example [9,10]).

References

[1] N. Aronszajn, T. Greese, L. Lipkin, Polyharmonic Functions, Clarendon Press, Oxford, 1983.
[2] B. Bojanov, An extension of the Pizzetti formula for polyharmonic functions, Acta Math. Hungar. 91 (2001) 99–113.
[3] B. Bojanov, Cubature formulae for polyharmonic functions, Recent Progress in Multivariate Approximation, (Witten Bommerholz, 2000), International Series of Numerical Mathematics, Vol. 137, Birkhäuser, Basel, 2001, pp. 49–74.
[4] B. Bojanov, D. Dimitrov, Gaussian extended cubature formulae for polyharmonic functions, Math. Comp. 70 (2000) 671–683.
[5] B. Bojanov, P. Petrov, Gaussian interval quadrature formula, Numer. Math. 87 (2001) 625–643.
[6] B. Bojanov, G. Petrova, Uniqueness of the Gaussian quadrature for a ball, J. Approx. Theory 104 (2000) 21–44.
[7] R. Cools, Ph. Rabinowitz, Monomial cubature rules since "Stroud": a compilation, J. Comput. Appl. Math. 48 (1993) 309–326.
[8] D. Dimitrov, Integration of polyharmonic functions, Math. Comp. 65 (1996) 1269–1281.
[9] M. Dubiner, The theory of multi-dimensional polynomial approximation, J. Anal. Math. 67 (1995) 39–116.
[10] M. Ganzburg, Polynomial approximation on the m-dimensional ball, Approximation Theory IX, Innov. Applied Mathematics, Vol. I, Vanderbilt Univ. Press, Nashville, TN 1998 pp. 141–148.
[11] M. Nicolescu, Les Fonctions Polyharmoniques, Hermann, Paris, 1936.
[12] M. Omladic, S. Pahor, A. Suhadolc, On a new type quadrature formulas, Numer. Math. 25 (4) (1976) 421–426.
[13] G. Petrova, Cubature formulae for the sphere and the ball in $\mathbb{R}^n$, in: B. Bojanov (Ed.), Proceedings volume of Constructive Function Theory, Varna 2002, DARBA, Sofia, 2003, pp. 380–384.
[14] G. Petrova, Uniqueness of the Gaussian extended cubature for polyharmonic functions, East. J. Approx., to appear.
[15] P. Pizzetti, Sulla media dei valore che una funzione dei punti della spazio assume allasuperticie di una sfera, Rend. Lincei, Ser. V, XVIII (1 sem) (1909) 182–185.
[16] M. Reimer, On the existence-problem for Gauss-quadrature on the sphere, Approximation by Solutions of Partial Differential Equations (Hanstholm, 1991), NATO Advanced Science Institutes Series C Mathematical and Physical Sciences, Vol. 365, Kluwer Academic Publishers, Dordrecht, 1992, pp. 169–184.
[17] A. Stroud, Approximate Calculation of Multiple Integrals, Prentice-Hall, Englewood Cliffs, NJ, 1971.
[18] Y. Xu, Orthogonal polynomials and cubature formulae on spheres and on simplices, Methods Appl. Anal. 5 (2) (1998) 169–184.